Sample Problem 3-2

A small airplane leaves an airport on an overcast day and is later sighted 215 km away, in a direction making an angle of 22° east of due north. How far east and north is the airplane from the airport when sighted?

KEY IDEA We are given the magnitude (215 km) and the angle (22° east of due north) of a vector and need to find the components of the vector.

Calculations: We draw an xy coordinate system with the positive direction of x due east and that of y due north (Fig. 3-10). For convenience, the origin is placed at the airport. The airplane's displacement $\vec{d}$ points from the origin to where the airplane is sighted.

To find the components of $\vec{d}$, we use Eq. 3-5 with $\theta = 68°$ $(= 90° - 22°)$:

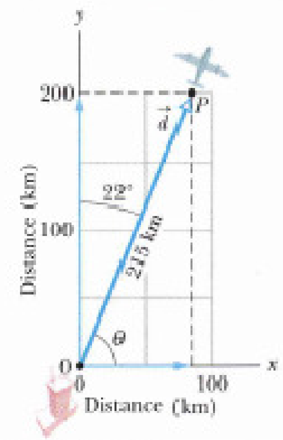

FIG. 3-10 A plane takes off from an airport at the origin and is later sighted at P.

$$d_x = d\cos\theta = (215 \text{ km})(\cos 68°) = 81 \text{ km} \quad \text{(Answer)}$$

$$d_y = d\sin\theta = (215 \text{ km})(\sin 68°) = 199 \text{ km} \approx 2.0 \times 10^2 \text{ km}. \quad \text{(Answer)}$$

Thus, the airplane is 81 km east and 2.0×10^2 km north of the airport.

Sample Problem 3-3

For two decades, spelunking teams sought a connection between the Flint Ridge cave system and Mammoth Cave, which are in Kentucky. When the connection was finally discovered, the combined system was declared the world's longest cave (more than 200 km long). The team that found the connection had to crawl, climb, and squirm through countless passages, traveling a net 2.6 km westward, 3.9 km southward, and 25 m upward. What was their displacement from start to finish?

KEY IDEA We have the components of a three-dimensional vector, and we need to find the vector's magnitude and two angles to specify the vector's direction.

Horizontal Components: We first draw the components as in Fig. 3-11a. The horizontal components (2.6 km west and 3.9 km south) form the legs of a horizontal right triangle. The team's horizontal displacement forms the hypotenuse of the triangle, and its magnitude d_h is given by the Pythagorean theorem:

$$d_h = \sqrt{(2.6 \text{ km})^2 + (3.9 \text{ km})^2} = 4.69 \text{ km}.$$

Also from the horizontal triangle in Fig. 3-11a, we see that this horizontal displacement is directed south of due west by an angle θ_h given by

$$\tan\theta_h = \frac{3.9 \text{ km}}{2.6 \text{ km}},$$

so

$$\theta_h = \tan^{-1}\frac{3.9 \text{ km}}{2.6 \text{ km}} = 56°, \quad \text{(Answer)}$$

which is one of the two angles we need to specify the direction of the overall displacement.

Overall Displacement: To include the vertical component (25 m = 0.025 km), we now take a side view of Fig. 3-11a, looking northwest. We get Fig. 3-11b, where the vertical component and the horizontal displacement d_h form the legs of another right triangle. Now the team's overall displacement forms the hypotenuse of that triangle, with a magnitude d given by

$$d = \sqrt{(4.69 \text{ km})^2 + (0.025 \text{ km})^2} = 4.69 \text{ km} \approx 4.7 \text{ km}. \quad \text{(Answer)}$$

This displacement is directed upward from the horizontal displacement by the angle

$$\theta_v = \tan^{-1}\frac{0.025 \text{ km}}{4.69 \text{ km}} = 0.3°. \quad \text{(Answer)}$$

Thus, the team's displacement vector had a magnitude of 4.7 km and was at an angle of 56° south of west and at an angle of 0.3° upward. The net vertical motion was, of course, insignificant compared with the horizontal motion. However, that fact would have been of no comfort to the team, which had to climb up and down countless times to get through the cave. The route they actually covered was quite different from the displacement vector.

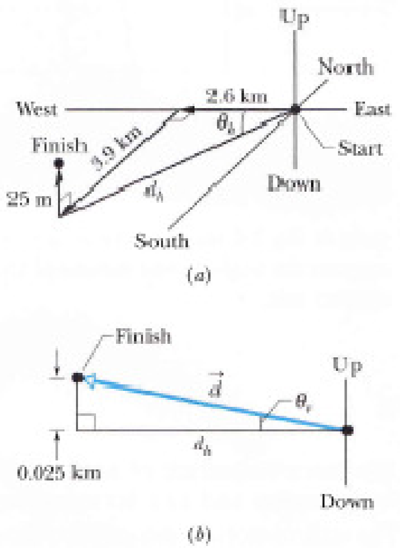

FIG. 3-11 (a) The components of the spelunking team's overall displacement and their horizontal displacement d_h. (b) A side view showing d_h and the team's overall displacement vector $\vec{d}$.

Sample Problem 3-4

Figure 3-16*a* shows the following three vectors:

$$\vec{a} = (4.2\text{ m})\hat{i} - (1.5\text{ m})\hat{j},$$
$$\vec{b} = (-1.6\text{ m})\hat{i} + (2.9\text{ m})\hat{j},$$
and
$$\vec{c} = (-3.7\text{ m})\hat{j}.$$

What is their vector sum $\vec{r}$ which is also shown?

KEY IDEA We can add the three vectors by components, axis by axis, and then combine the components to write the vector sum $\vec{r}$.

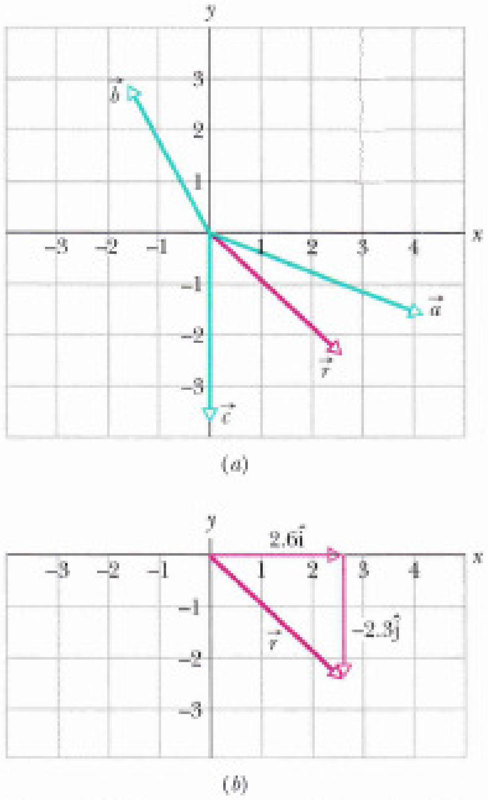

FIG. 3-16 Vector $\vec{r}$ is the vector sum of the other three vectors.

Calculations: For the x axis, we add the x components of $\vec{a}$, $\vec{b}$, and $\vec{c}$, to get the x component of the vector sum $\vec{r}$:

$$r_x = a_x + b_x + c_x$$
$$= 4.2\text{ m} - 1.6\text{ m} + 0 = 2.6\text{ m}.$$

Similarly, for the y axis,

$$r_y = a_y + b_y + c_y$$
$$= -1.5\text{ m} + 2.9\text{ m} - 3.7\text{ m} = -2.3\text{ m}.$$

We then combine these components of $\vec{r}$ to write the vector in unit-vector notation:

$$\vec{r} = (2.6\text{ m})\hat{i} - (2.3\text{ m})\hat{j}, \qquad \text{(Answer)}$$

where $(2.6\text{ m})\hat{i}$ is the vector component of $\vec{r}$ along the x axis and $-(2.3\text{ m})\hat{j}$ is that along the y axis. Figure 3-16*b* shows one way to arrange these vector components to form $\vec{r}$. (Can you sketch the other way?)

We can also answer the question by giving the magnitude and an angle for $\vec{r}$. From Eq. 3-6, the magnitude is

$$r = \sqrt{(2.6\text{ m})^2 + (-2.3\text{ m})^2} \approx 3.5\text{ m} \qquad \text{(Answer)}$$

and the angle (measured from the $+x$ direction) is

$$\theta = \tan^{-1}\left(\frac{-2.3\text{ m}}{2.6\text{ m}}\right) = -41°, \qquad \text{(Answer)}$$

where the minus sign means clockwise.

Sample Problem 3-6 Build your skill

Here is a problem involving vector addition that *cannot* be solved directly on a vector-capable calculator, using the vector notation of the calculator. A fellow camper is to walk away from you in a straight line (vector $\vec{A}$), turn, walk in a second straight line (vector $\vec{B}$) and then stop. How far must you walk in a straight line (vector $\vec{C}$) to reach her?

The three vectors (shown in Fig. 3-18) are related by

$$\vec{C} = \vec{A} + \vec{B}. \qquad (3\text{-}16)$$

$\vec{A}$ has a magnitude of 22.0 m and is directed at an angle of −47.0° (clockwise) from the positive direction of an x axis. $\vec{B}$ has a magnitude of 17.0 m and is directed counterclockwise from the positive direction of the x axis by angle ϕ. $\vec{C}$ is in the positive direction of the x axis. What is the magnitude of $\vec{C}$?

KEY IDEA We cannot answer the question by adding $\vec{A}$ and $\vec{B}$ directly on a vector-capable calculator, say, in the generic form of

[magnitude A ∠ angle A] + [magnitude B ∠ angle B]

because we do not know the value for the angle ϕ of $\vec{B}$. However, we *can* express Eq. 3-16 in terms of components for either the x axis or the y axis.

Calculations: Since $\vec{C}$ is directed along the x axis, we choose that axis and write

$$C_x = A_x + B_x.$$

We next express each x component in the form of the x part of Eq. 3-5 and substitute known data. We then have

$$C\cos 0° = 22.0\cos(-47.0°) + 17.0\cos\phi. \qquad (3\text{-}17)$$

However, this hardly seems to help, because we still cannot solve for C without knowing ϕ.

Let us now express Eq. 3-16 in terms of components along the y axis:

$$C_y = A_y + B_y.$$

We then cast these y components in the form of the y part of Eq. 3-5 and substitute known data, to write

$$C\sin 0° = 22.0\sin(-47.0°) + 17.0\sin\phi,$$

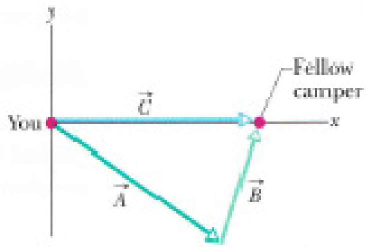

FIG. 3-18 $\vec{C}$ equals the sum $\vec{A} + \vec{B}$.

which yields

$$0 = 22.0 \sin(-47.0°) + 17.0 \sin \phi.$$

Solving for ϕ then gives us

$$\phi = \sin^{-1} - \frac{22.0 \sin(-47.0°)}{17.0} = 71.17°.$$

Substituting this result into Eq. 3-17 leads us to

$$C = 20.5 \text{ m}. \qquad \text{(Answer)}$$

Note the technique of solution: When we got stuck with components on the x axis, we worked with components on the y axis, to evaluate ϕ. We next moved back to the x axis, to evaluate C.

Sample Problem 3-7

What is the angle ϕ between $\vec{a} = 3.0\hat{i} - 4.0\hat{j}$ and $\vec{b} = -2.0\hat{i} + 3.0\hat{k}$? (*Caution:* Although many of the following steps can be bypassed with a vector-capable calculator, you will learn more about scalar products if, at least here, you use these steps.)

KEY IDEA The angle between the directions of two vectors is included in the definition of their scalar product (Eq. 3-20):

$$\vec{a} \cdot \vec{b} = ab \cos \phi. \qquad (3\text{-}24)$$

Calculations: In Eq. 3-24, a is the magnitude of $\vec{a}$, or

We next apply Eq. 3-20 to each term in this last expression. The angle between the unit vectors in the first term ($\hat{i}$ and $\hat{i}$) is 0°, and in the other terms it is 90°. We then have

$$\vec{a} \cdot \vec{b} = -(6.0)(1) + (9.0)(0) + (8.0)(0) - (12)(0)$$
$$= -6.0.$$

We next apply Eq. 3-20 to each term in this last expression. The angle between the unit vectors in the first term ($\hat{i}$ and $\hat{i}$) is 0°, and in the other terms it is 90°. We then have

$$\vec{a} \cdot \vec{b} = -(6.0)(1) + (9.0)(0) + (8.0)(0) - (12)(0)$$
$$= -6.0.$$

$$a = \sqrt{3.0^2 + (-4.0)^2} = 5.00, \qquad (3\text{-}25)$$

and b is the magnitude of $\vec{b}$, or

$$b = \sqrt{(-2.0)^2 + 3.0^2} = 3.61. \qquad (3\text{-}26)$$

We can separately evaluate the left side of Eq. 3-24 by writing the vectors in unit-vector notation and using the distributive law:

$$\vec{a} \cdot \vec{b} = (3.0\hat{i} - 4.0\hat{j}) \cdot (-2.0\hat{i} + 3.0\hat{k})$$
$$= (3.0\hat{i}) \cdot (-2.0\hat{i}) + (3.0\hat{i}) \cdot (3.0\hat{k})$$
$$+ (-4.0\hat{j}) \cdot (-2.0\hat{i}) + (-4.0\hat{j}) \cdot (3.0\hat{k}).$$

Substituting this result and the results of Eqs. 3-25 and 3-26 into Eq. 3-24 yields

$$-6.0 = (5.00)(3.61) \cos \phi,$$

so $$\phi = \cos^{-1} \frac{-6.0}{(5.00)(3.61)} = 109° \approx 110°. \qquad \text{(Answer)}$$

Substituting this result and the results of Eqs. 3-25 and 3-26 into Eq. 3-24 yields

$$-6.0 = (5.00)(3.61) \cos \phi,$$

so $$\phi = \cos^{-1} \frac{-6.0}{(5.00)(3.61)} = 109° \approx 110°. \qquad \text{(Answer)}$$

Sample Problem 3-9

If $\vec{a} = 3\hat{i} - 4\hat{j}$ and $\vec{b} = -2\hat{i} + 3\hat{k}$, what is $\vec{c} = \vec{a} \times \vec{b}$?

KEY IDEA When two vectors are in unit-vector notation, we can find their cross product by using the distributive law.

Calculations: Here we write

$$\vec{c} = (3\hat{i} - 4\hat{j}) \times (-2\hat{i} + 3\hat{k})$$
$$= 3\hat{i} \times (-2\hat{i}) + 3\hat{i} \times 3\hat{k} + (-4\hat{j}) \times (-2\hat{i})$$
$$+ (-4\hat{j}) \times 3\hat{k}.$$

We next evaluate each term with Eq. 3-27, finding the direction with the right-hand rule. For the first term here, the angle ϕ between the two vectors being crossed is 0. For the other terms, ϕ is 90°. We find

$$\vec{c} = -6(0) + 9(-\hat{j}) + 8(-\hat{k}) - 12\hat{i}$$
$$= -12\hat{i} - 9\hat{j} - 8\hat{k}. \qquad \text{(Answer)}$$

This vector $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, a fact you can check by showing that $\vec{c} \cdot \vec{a} = 0$ and $\vec{c} \cdot \vec{b} = 0$; that is, there is no component of $\vec{c}$ along the direction of either $\vec{a}$ or $\vec{b}$.